

Optimal control of filtration and back-washing under membrane clogging

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Abstract

This paper presents an optimal control strategy allowing the maximization of the total production of a membrane filtration system over a finite time horizon. A simple mathematical model of membrane fouling is used to capture the dynamic behavior of the process which consists in the attachment of matter onto the membrane during the filtration period and the detachment of matter during the cleaning period. The control variable is the sequence of filtration/back-washing cycles over the time. Based on the Pontryagin's Maximum Principle, we establish an optimal control strategy involving a singular arc and a switching curve.

Key-words. Optimal control, singular arc, membrane fouling, maximization of water production, MBR, backwash.

1 Introduction

Membrane filtration systems are widely used as physical separation techniques in different industrial fields like water desalination, waste-water treatment, food, medicine and biotechnology. The membrane provides a selective barrier that separates substances when a driving force is applied across the membrane.

The main disadvantages of these processes is the membrane fouling by the continuous accumulation of the filtered impurities onto the membrane surface (filter cake) and pores. Different fouling mechanisms are responsible of the flux decline at constant trans-membrane pressure (TMP) or the increase of the TMP at a constant flux. Hence, the operation of the membrane filtering process requires to perform regularly cleaning actions like relaxation, aeration, back-washing and chemical cleaning to limit the membrane fouling and maintain a good water production.

Usually, sequences of filtration and membrane cleaning are fixed according to the recommendations of the membrane suppliers or chosen according to the operator's experience. This leads to high operational cost and to performances (quantities of water filtered over a given period of time) that can be far from being optimal. For this reason, it is important to optimize the membrane filtration process functioning in order to maximize system performances while minimizing energy costs.

A variety of control approaches have been proposed to manage filtration processes. In practice such strategies are based on the application of a cleaning action (physical or chemical) when either the flux decline through the membrane or the TMP increase crosses predefined threshold values ([6]). [12] developed a control system that monitors the TMP evolution over time and initiates a membrane backwash

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when the TMP exceeds a given set-point. [8] use also the TMP as the monitoring variable but the control action was the increase or decrease of membrane aeration. [13] use the permeate flux as the controlled variable to optimize the membrane back-washing and prevent fouling. On the other hand, knowledge-based controllers find application in the control of membrane filtration process. [11] have proposed an advanced control system composed of a knowledge-based controller and two classical controller (on/off and PID) to manage the aeration and backwash sequences. Finally, the permeability was used by [5] as a monitoring variable in a knowledge-based control system to control membrane aeration flow.

To date, different available control systems are able to increase significantly the membrane filtration process performances. However, more enhanced optimizing control strategies are needed to cope with the dynamic operation of the purifying system and to limit membrane fouling. The majority of the control strategies previously cited address energy consumption. In the present work, we are interested in maximizing the water production of a membrane filtration system over a given time period T by optimizing the ratio of filtration/back-washing time-periods within a given number of filtration/cleaning sequence (the backwash is a period during which there is no filtration: during this period, the material attached onto the membrane can then naturally detach).

To describe the membrane filtration process, we consider a simple form of the model of [1]. In a previous work, it was shown that this model is very generic in the sense that it is able to capture the dynamics of a large number of models available in the literature and simple enough to be used for optimizing and control purposes, cf. [7]. In the present work, it is assumed that the membrane fouling is only due to the particle deposition onto the membrane surface.

2 Model description and preliminaries

Let m be the mass of the membrane deposit during the water filtration. One can assume that m is growing according to a dynamics $\dot{m} = f_1(m)$. During a wash-out, the flow is reversed and the mass of the *cake* accumulated on the membrane is reduced according to a dynamics $\dot{m} = -f_2(m)$. We consider a control u that takes values 1 during filtration and -1 during retro washing. Then, the controlled dynamics can be written as follows

$$\dot{m} = \frac{1+u}{2}f_1(m) - \frac{1-u}{2}f_2(m), \quad m(0) = m_0. \quad (1)$$

Assuming that the water flow that passes through the membrane is given by a function g that depends on m , the total amount of water that is treated by the membrane during a time interval $[0, T]$ is then

$$J_T(m_0, u(\cdot)) := \int_0^T u(t)g(m(t))dt.$$

Given an initial mass $m_0 > 0$, the objective is to determine an optimal strategy $u(\cdot)$ that takes values -1 or 1 for maximizing $J_T(m_0, u(\cdot))$. Nevertheless, it is well known from the Theory of Optimal Control that the existence of an optimal trajectory can be guaranteed when the control set is convex [9]. Therefore, we shall consider for the mathematical analysis that the control $u(\cdot)$ can takes values in the interval $[-1, 1]$. The question of practical applicability of a control that takes values different to -1 and 1 (by approximation) will be the matter of a future work.

Assumption 2.1. *The functions f_1 , f_2 and g are C^1 functions such that*

- i. $f_1(m) > 0$ and $g(m) > 0$ for any $m \geq 0$*
- ii. $f_2(0) = 0$ and $f_2(m) > 0$ for $m > 0$*
- iii. f_1 and g are decreasing with $\lim_{m \rightarrow +\infty} g(m) = 0$*
- iv. f_2 is increasing*

One can straightforwardly check the following property

Lemma 2.1. *Under Assumption 2.1, the domain $\{m > 0\}$ is positively invariant whatever is the control $u(\cdot)$.*

For convenience, we define

$$f_+(m) = \frac{f_1(m) + f_2(m)}{2}, \quad f_-(m) = \frac{f_1(m) - f_2(m)}{2}$$

We shall use the Maximum Principle of Pontryagin (PMP) [10] in order to determine necessary conditions on optimal trajectories. For this we introduce the Hamiltonian of the system

$$H(m, \lambda, u) = \lambda f_-(m) + u(\lambda f_+(m) + g(m)) \quad (2)$$

to be maximized w.r.t. u at u^* such that

$$u^* = \begin{cases} 1 & \text{when } \phi(m, \lambda) > 0 \\ -1 & \text{when } \phi(m, \lambda) < 0 \end{cases}$$

where ϕ is the switching function

$$\phi(m, \lambda) := \lambda f_+(m) + g(m)$$

The adjoint equation given by the PMP is then

$$\dot{\lambda} = -\partial_m H(m, \lambda, u^*) = -\lambda f'_-(m) - u^* (\lambda f'_+(m) + g'(m)) \quad (3)$$

with the terminal condition $\lambda(T) = 0$.

Proposition 2.1. *Under Assumption 2.1, the adjoint variable satisfies $\lambda(T) < 0$ for any $t \in [0, T[$. moreover, for any initial condition m_0 there exists $\bar{t} < T$ such that $u(t) = 1$ is optimal for $t \in [\bar{t}, T]$.*

Proof. At $\lambda = 0$, one has $\phi(m, 0) = g(m) > 0$ and then $u^* = 1$ which implies to have $\dot{\lambda} = -g'(m) > 0$. If $\lambda(t) = 0$ for some $\bar{t} < T$ then one has necessarily $\lambda(t) > 0$ for any $t > \bar{t}$ which is in contradiction with $\lambda(T) = 0$. Therefore $t \mapsto \lambda(t)$ is non-null and has constant sign on $[0, T[$. As λ has to reach 0 at time T with $\lambda(T) > 0$, we conclude that λ has to be negative on $[0, T[$.

At terminal time, one has $\phi(m(T), \lambda(T)) = \phi(m(T), 0) = g(m(T)) > 0$. By continuity, the function $t \mapsto \phi(m(t), \lambda(t))$ is positive on a time interval $[\bar{t}, T]$ with $\bar{t} < T$, thus the optimality of $u = 1$ on this interval. \square

3 Study of the singular arc

For convenience, we define the function

$$\psi(m) = g(m) (f'_-(m)f_+(m) - f_-(m)f'_+(m)) + g'(m)f_+(m)f_-(m)$$

and consider the following hypothesis

Assumption 3.1. *The function ψ admits an unique positive root \bar{m} and is such that $\psi(m)(m - \bar{m}) > 0$ for any positive $m \neq \bar{m}$.*

Under this condition, one can characterize $m = \bar{m}$ as the unique candidate singular arc, along with the following properties of the optimal trajectories (see [2] for a thorough study of this notion).

Proposition 3.1. *Under Assumptions 2.1 and 3.1, one has the following properties:*

- i. when $m_0 < \bar{m}$, $u = 1$ is optimal as long as $m(t) < \bar{m}$,
- ii. when $m_0 > \bar{m}$, either $u = 1$ is optimal until $t = T$, either $u = -1$ is optimal until a time $\bar{t} < T$ with $m(\bar{t}) \geq \bar{m}$. If $m(\bar{t}) > \bar{m}$ or $f_-(\bar{m}) \geq 0$ then $u = 1$ is optimal on $[\bar{t}, T]$
- iii. if $f_-(\bar{m}) < 0$, define

$$\bar{T} = T - \int_{\bar{m}}^{\bar{m}_T} \frac{dm}{f_1(m)} \quad \text{with} \quad \bar{m}_T = g^{-1} \left(-g(\bar{m}) \frac{f_-(\bar{m})}{f_+(\bar{m})} \right) \quad (4)$$

If $m(t) = \bar{m}$ with $t < \bar{T}$ then the singular arc $m = \bar{m}$ is optimal until \bar{T} , with the constant control

$$\bar{u} = -\frac{f_-(\bar{m})}{f_+(\bar{m})}. \quad (5)$$

If $m(t) \geq \bar{m}$ with $t \geq \bar{T}$, then $u = 1$ is optimal until T .

Proof. Let us write the derivative of the switching function (we drop the m dependency of functions f_- , f_+ and g for simplicity):

$$\begin{aligned}\dot{\phi} &= -(\lambda f'_- + u(\lambda f'_+ + g')) + (\lambda f'_+ + g')(f_- + f_+ u) \\ &= \lambda(f'_+ f_- - f'_- f_+) + g' f_- \\ &= g(f'_- - f'_+ f_- / f_+) + g' f_- + \phi \frac{f'_+ f_- - f'_- f_+}{f_+}\end{aligned}$$

or equivalently

$$\dot{\phi} = \frac{\psi}{f_+} + \phi \frac{f'_+ f_- - f'_- f_+}{f_+} \quad (6)$$

As a singular arc has to fulfill $\phi = 0$ and $\dot{\phi} = 0$, equation (6) and Assumption 3.1 gives $\psi = 0$. Then, the single possibility for having a singular arc on a time interval $[t_1, t_2]$ is to have $m(t) = \bar{m}$ for any $t \in [t_1, t_2]$. From equation (1), one then obtains the constant control given in (5) for having $\dot{m} = 0$ at $m = \bar{m}$. From Assumption 3.1 and equation (6), the following properties are then satisfied

- $\phi = 0$ with $m < \bar{m} \Rightarrow \dot{\phi} < 0$. This implies that ϕ can change its sign only when decreasing. Therefore only a switch $u = 1$ to $u = -1$ can be optimal in the domain $\{m < \bar{m}\}$.
- $\phi = 0$ with $m > \bar{m} \Rightarrow \dot{\phi} > 0$. This implies that ϕ can change its sign only when increasing. Therefore only a switch $u = -1$ to $u = 1$ can be optimal in the domain $\{m > \bar{m}\}$.

At $m = \bar{m}$, $u = -1$ cannot be optimal. Otherwise, $\dot{m} < 0$ and m enters the domain $\{m < \bar{m}\}$ with $u = -1$. Then one has $m < \bar{m}$ and $u = -1$ for any future time (as a switch $u = -1$ to $u = 1$ cannot be optimal on this domain), which contradicts Proposition 2.1.

If it is optimal to stay on the singular arc $m = \bar{m}$ on a time interval of non-null length, then the adjoint variable has to be constant $\lambda = \bar{\lambda}$ on this time interval, to guarantee $\phi = 0$, which amounts to have:

$$\bar{\lambda} = -\frac{g(\bar{m})}{f_+(\bar{m})}$$

As the Hamiltonian is constant along any optimal trajectory, one has to have $H = \bar{\lambda} f_-(\bar{m})$. Moreover, as the Hamiltonian at time T is given by $H = g(m(T))$, one should have $\bar{\lambda} f_-(\bar{m}) = g(m(T)) > 0$. As $\bar{\lambda} < 0$, we conclude that when $f_-(\bar{m}) \geq 0$ a singular arc cannot be optimal.

Consider now the case $f_-(\bar{m}) > 0$. Accordingly to Propositions 2.1 and 3.1, an optimal trajectory has to leave the singular arc with $u = 1$ at a certain switching time $\bar{T} < T$ until reaching terminal time T . This imposes the final state to be $\bar{m}_T = m(T)$ as a solution of

$$g(\bar{m}_T) = \bar{\lambda} f_-(\bar{m}) = -\frac{g(\bar{m}) f_-(\bar{m})}{f_+(\bar{m})}, \quad (7)$$

which is uniquely defined as g is decreasing and such that $g(+\infty) = 0$. This also imposes the switching time \bar{T} that can be determined integrating backward the system

$$\dot{m} = f_1(m), \quad m(T) = \bar{m}_T$$

until $m(\bar{T}) = \bar{m}$, which amounts to write

$$\bar{T} = T - \int_{\bar{m}}^{\bar{m}_T} \frac{dm}{f_1(m)}. \quad (8)$$

We show now when $m(t) = \bar{m}$ with $t < \bar{T}$ then it is optimal to stay on the singular arc until \bar{T} (and then use $u = 1$ from \bar{T} to T). If not, accordingly to the properties proved above, the only possibility to leave the singular arc is to use the control $u = 1$ until T , and thus to have $m(T) > \bar{m}_T$. As the dynamics is $\dot{m} = f_1(m)$ with such a control, the corresponding cost from time t can be written as follows:

$$J_1(t) = \int_{\bar{m}}^{m(T)} \frac{g(m)}{f_1(m)} dm,$$

to be compared with the cost of the singular arc strategy, which is

$$J_s(t) = -\frac{g(\bar{m})f_-(\bar{m})}{f_+(\bar{m})}(\bar{T} - t) + \int_{\bar{m}}^{\bar{m}_T} \frac{g(m)}{f_1(m)} dm.$$

With the expression(8), one can write

$$\bar{T} - t = (T - t) - \int_{\bar{m}}^{\bar{m}_T} \frac{dm}{f_1(m)} = \int_{\bar{m}}^{m(T)} \frac{dm}{f_1(m)} - \int_{\bar{m}}^{\bar{m}_T} \frac{dm}{f_1(m)} = \int_{\bar{m}_T}^{m(T)} \frac{dm}{f_1(m)}.$$

Then, one can consider the difference of costs

$$\delta(m(T)) := J_1(t) - J_s(t) = \int_{\bar{m}_T}^{m(T)} \left(g(m) + \frac{g(\bar{m})f_-(\bar{m})}{f_+(\bar{m})} \right) \frac{dm}{f_1(m)}$$

and compute the two first derivatives of the function δ :

$$\begin{aligned} \delta'(m) &= \left(g(m) + \frac{g(\bar{m})f_-(\bar{m})}{f_+(\bar{m})} \right) \frac{1}{f_1(m)} \\ \delta''(m) &= \frac{g'(m) - \delta'(m)f_1'(m)}{f_1(m)} \end{aligned}$$

From this last expression, one has

$$\delta'(m) = 0 \implies \delta''(m) < 0$$

and as $\delta'(m_T) = 0$ (from equation (7)) we deduce that $\delta'(m) < 0$ for any $m > m_T$. Finally, as $\delta(m_T) = 0$ we obtain $\delta(m(T)) < 0$. □

This first result allows to give the optimal synthesis in the domain

$$\mathcal{D}_- := \{(t, m) \in [0, T] \times [0, \bar{m}]\}.$$

Proposition 3.2. *If $f_-(\bar{m}) \geq 0$ or $\bar{T} \leq 0$ (where \bar{T} is defined in (4) when $f_-(\bar{m}) < 0$), then $u = 1$ is optimal at any $(t, x) \in \mathcal{D}_-$. Otherwise, the strategy*

$$u^*(t, x) = \begin{cases} 1 & \text{if } m < \bar{m} \text{ or } t \geq \bar{T}, \\ \bar{u} & \text{if } m = \bar{m} \text{ and } t < \bar{T} \end{cases}$$

is optimal at any $(t, x) \in \mathcal{D}_-$.

Proposition 3.1 allows also to state that the domain

$$\mathcal{D}_+ := \{(t, m) \in [0, T] \times [\bar{m}, +\infty)\}$$

is optimally invariant:

Corollary 3.1. *From any $(t, m) \in \mathcal{D}_+$, the optimal trajectory stays in \mathcal{D}_+ for any future time.*

4 Study of the switching curve

Accordingly to Proposition 3.1, there is a possibility to have a switch $u = -1$ to $u = 1$ for an optimal trajectory in the domain \mathcal{D}_+ . The following proposition gives existence and characterization of a switching curve in this domain. For convenience, we define the function

$$\gamma(m) = -\frac{g(m)f_-(m)}{f_+(m)}, \quad m \geq \bar{m}.$$

Proposition 4.1. *Assume that Hypotheses 2.1 and 3.1 are fulfilled.*

- If $f_-(\bar{m}) \geq 0$, then the optimal control is $u = 1$ for any initial condition in \mathcal{D}_+ .
- If $f_-(\bar{m}) < 0$, consider the (possibly empty) set

$$\mathcal{C} := \left\{ (\tilde{T}(\tilde{m}), \tilde{m}) \mid \tilde{m} \geq \bar{m} \text{ and } \tilde{T}(\tilde{m}) > 0 \right\}$$

where

$$\tilde{T}(\tilde{m}) = T - \int_{\tilde{m}}^{g^{-1}(\gamma(\tilde{m}))} \frac{dm}{f_1(m)}, \quad \tilde{m} \geq \bar{m}$$

If \mathcal{C} is empty, then the optimal control is $u = 1$ for any initial condition in \mathcal{D}_+ .

If \mathcal{C} is non empty, then the strategy

$$u(t, m) = \begin{cases} 1 & \text{if } (t, m) \text{ is above } \mathcal{C} \text{ or } t \geq \tilde{T} \\ 0 & \text{if } m > \bar{m} \text{ and } (t, m) \text{ is below } \mathcal{C} \\ \bar{u} & \text{if } m = \bar{m} \text{ and } t < \tilde{T} \end{cases}$$

is optimal for any $(t, m) \in \mathcal{D}_+$. Furthermore, the curve \mathcal{C} is tangent to the trajectory that leaves the singular arc at (\tilde{T}, \bar{m}) with the control $u = 1$.

Proof. We know from Proposition 2.1 that an optimal trajectory reaches the final time with the control $u = 1$. Let $m_T = m(T)$ be the terminal state of an optimal trajectory.

When $f_-(\bar{m}) \geq 0$, we already know that the optimal control is $u = 1$ at any time if $m_T < \bar{m}$, as the trajectory stays in the domain \mathcal{D}_- .

When $f_-(\bar{m}) > 0$ and $m_T < \bar{m}_T$ (where \bar{m}_T is defined in (7)), the trajectory could not have crossed $m = \bar{m}$ before \tilde{T} (by uniqueness of the solution of the dynamics $\dot{m} = f_1(m)$ with the control $u = 1$) and no switching is possible for such optimal trajectories i.e. the constant control $u = 1$ is optimal.

We consider now terminal states $m_T \geq \bar{m}$ (when $f_-(\bar{m}) \geq 0$) or $m_T \geq \bar{m}_T$ (when $f_-(\bar{m}) < 0$), and integrate backward the dynamics with the control $u = 1$. One has $H = g(m_T) = g(m(t)) + \lambda(t)f_1(m(t))$ for $t < T$ as long as the switching function

$$\begin{aligned} \phi(m, \lambda) &= g(m) + \lambda f_+(m) \\ &= g(m) + (g(m_T) - g(m)) \frac{f_+(m)}{f_1(m)} \\ &= \frac{f_+(m)}{f_1(m)} (g(m_T) - \gamma(m)) \end{aligned}$$

is positive.

When $f_-(m) \geq 0$, one has $f_-(m(t)) \geq 0$ for any time t as $m(t) \leq \bar{m}$ and $f_-(\cdot)$ is decreasing. Then one has $\gamma(m(t)) \leq 0$ at any time and ϕ cannot change its sign. Therefore $u = 1$ is optimal at any time.

When $f_-(m) < 0$, notice that for $m_T = \bar{m}_T$, one has $g(\bar{m}_T) = \psi(\bar{m})$ (g is decreasing), and that for any $m_T \geq \bar{m}_T$, $m = \bar{m}$ is reached backward in time ($f_1(m)$ is strictly positive). Then for $m_T > \bar{m}_T$, one has $\phi < 0$ at $m = \bar{m}$. By the mean value theorem, we conclude that a switch necessary occurs at a $\tilde{m} > \bar{m}$ such that $\gamma(\tilde{m}) = g(m_T)$, and accordingly to Proposition 3.1 this switch ($u = -1$ to $u = 1$) is unique along optimal trajectories such that terminal state satisfies $m(T) = m_T > \bar{m}_T$.

Compute the derivative of the function γ :

$$\gamma' = -\frac{g'f_- + gf_-'}{f_+} + \frac{gf_-f_+'}{f_+^2} = -\frac{\psi}{(f_1 + f_2)^2}. \quad (9)$$

Then, by Assumption 3.1 one has $\gamma'(m) < 0$ for $m > \bar{m}$. Therefore, γ is invertible for $m > \bar{m}$ and \tilde{m} is uniquely defined as $\tilde{m} = \gamma^{-1}(g(m_T))$, or reciprocally, for any $\tilde{m} \geq \bar{m}$, m_T is uniquely defined as a function of \tilde{m} : $m_T(\tilde{m}) = g^{-1}(\gamma(\tilde{m}))$ (as g is also a decreasing invertible function), with

$$m_T'(\tilde{m}) = \frac{\gamma'(\tilde{m})}{g'(\tilde{m})} \geq 0. \quad (10)$$

Then, the corresponding switching time $\tilde{T}(\tilde{m})$ satisfies

$$T - \tilde{T}(\tilde{m}) = \int_{\tilde{m}}^{m_T(\tilde{m})} \frac{dm}{f_1(m)}. \quad (11)$$

If $\tilde{T}(\tilde{m}) \leq 0$ then no switch occurs at \tilde{m} i.e. the constant control $u = 1$ is optimal from 0 to T .

The derivative of \tilde{T} with respect to \tilde{m} can be determined from expressions (11) and (10) as

$$\tilde{T}'(\tilde{m}) = \frac{1}{f_1(\tilde{m})} - \frac{m'_T(\tilde{m})}{f_1(m_T(\tilde{m}))} = \frac{1}{f_1(\tilde{m})} - \frac{\gamma'(\tilde{m})}{g'(m_T(\tilde{m}))f_1(m_T(\tilde{m}))}.$$

At $\tilde{m} = \bar{m}$, one has $\tilde{T}(\bar{m}) = \bar{T}$ and $\gamma'(\bar{m}) = 0$ (from (9)), which gives $\tilde{T}'(\bar{m}) = 1/f_1(\bar{m})$. Thus, the curve \mathcal{C} is tangent to the trajectory that leaves the singular arc with $u = 1$ at (T, \bar{m}) . \square

5 Application

5.1 Benyahia et al model

Consider the following functions that have validated on experimental data [1]

$$f_1(m) = \frac{b}{e+m}, \quad f_2(m) = am, \quad g(m) = \frac{1}{e+m}$$

where a , b and e are positive numbers. One can check that Assumption 2.1 is fulfilled. A straightforward computation of the function ψ gives

$$\begin{aligned} \psi(m) &= \frac{-1}{2(e+m)} \left[\left(\frac{b}{(e+m)^2} + a \right) \left(\frac{b}{e+m} + am \right) + \left(\frac{b}{e+m} - am \right) \left(\frac{b}{(e+m)^2} - a \right) \right] \\ &\quad - \frac{1}{2(e+m)^2} \left(\frac{b}{e+m} + am \right) \left(\frac{b}{e+m} - am \right) \\ &= \frac{a^2 e^2 m^2 + 2 a^2 e m^3 + a^2 m^4 - 2 a b e^2 - 6 a b e m - 4 a b m^2 - b^2}{4(e+m)^4} \end{aligned}$$

A further computation of the derivative of ψ gives

$$\psi'(m) = \frac{a^2 e^3 m + 2 a^2 e^2 m^2 + a^2 e m^3 + a b e^2 + 5 a b e m + 4 a b m^2 + 2 b^2}{2(e+m)^5}$$

which allows to conclude that ψ is increasing on \mathbb{R}_+ . As one has $\psi(0) = -(2abe^2 + b^2)/(4e^4) < 0$ and $\lim_{m \rightarrow +\infty} \psi(m) = +\infty$, we deduce that hypothesis 3.1 is fulfilled. When ψ is null for $m = \bar{m}$, one has

$$d(\bar{m}) = f'_-(\bar{m})f_+(\bar{m}) - f_-(\bar{m})f'_+(\bar{m}) = \frac{-g'(\bar{m})f_+(\bar{m})}{g(\bar{m})}f_-(\bar{m})$$

Therefore $f_-(\bar{m})$ and $d(\bar{m})$ have the same sign. A straightforward computation gives

$$d(m) = -\frac{ab(e+2m)}{2(e+m)^2} < 0$$

and thus one has $d(\bar{m}) < 0$. Therefore, from Propositions 3.1 and 4.1, there exists a singular arc when $\bar{T} > 0$ and a switching curve when $\tilde{T}(\tilde{m}) > 0$.

Figure 1 shows the general synthesis of the optimal control with the parameters $a = b = e = 1$ and for a time horizon of 10 hours.

5.2 Cogan-Chellam model

We consider the functions

$$f_1(m) = \frac{b}{e+m}, \quad f_2(m) = \frac{am}{e+m}, \quad g(m) = \frac{1}{e+m}$$

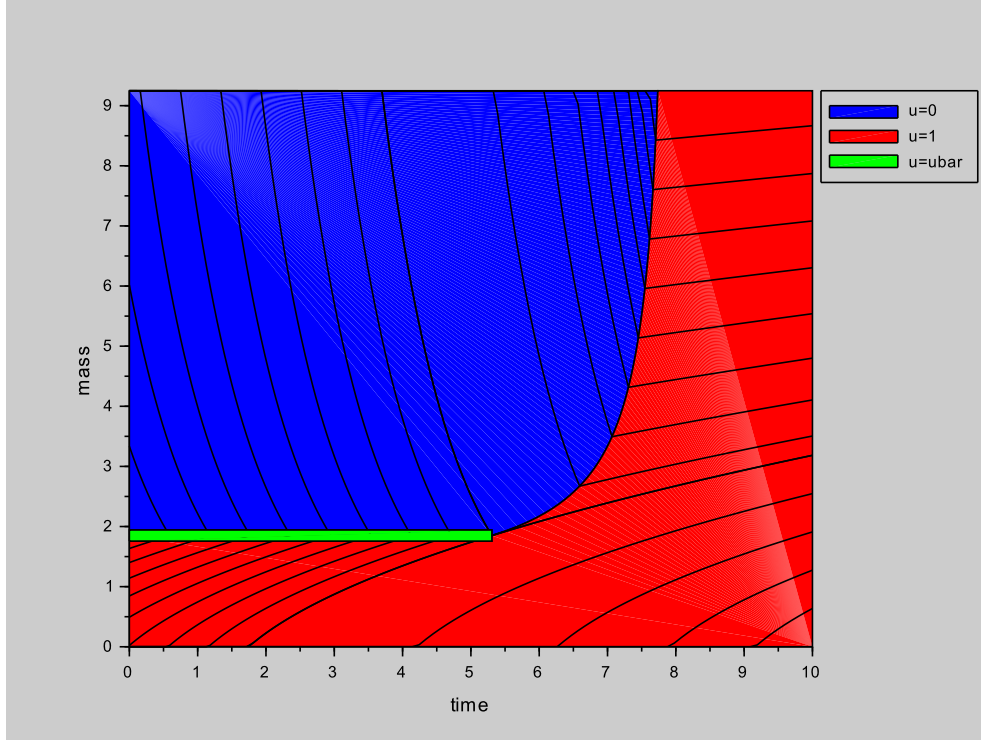


Figure 1: Model of Section 5.1: Synthesis for $a = b = e = 1$ and $T = 10$ hours; in red, $u = 1$, in blue $u = 0$, the horizontal line which separates them corresponds to \bar{u} while the curve connected to this line and which separates the red and blue regions on the right of the figure is the “switching curve”

where a , b and e are positive numbers, as proposed in [3, 4]. Clearly Assumption 2.1 is fulfilled. One has

$$\begin{aligned}\psi(m) &= -\frac{(ae+b)(b+am) + (ae-b)(b-am)}{4(e+m)^4} - \frac{(b+am)(b-am)}{4(e+m)^4} \\ &= \frac{a^2m^2 - 2abe - 2abm - 2b^2}{4(e+m)^4} = \frac{(am-b)^2 - 2abe - b^2}{4(e+m)^4}\end{aligned}$$

Therefore, the function ψ can have two changes of sign at

$$\bar{m}_- = \frac{b - \sqrt{b^2 + 2abe}}{a}, \quad \bar{m}_+ = \frac{b + \sqrt{b^2 + 2abe}}{a}$$

\bar{m}_- and \bar{m}_+ are respectively negative and positive numbers. One has also

$$\psi'(m) = \frac{a^2em + abe + abm}{2(e+m)^5}$$

which is positive. Therefore ψ is an increasing function and Hypothesis 3.1 is fulfilled with $\bar{m} = \bar{m}_+$. Moreover one can write

$$f_-(\bar{m}) = \frac{-\sqrt{b^2 + 2abe}}{e + \bar{m}} < 0.$$

Then, as for the previous model, Propositions 3.1 and 4.1 allow to conclude that there exists a singular arc when $\bar{T} > 0$ and a switching curve when $\tilde{T}(\tilde{m}) > 0$.

Figure 2 shows the synthesis of the optimal for this model $a = b = e = 1$ and for a time horizon of 40 hours.

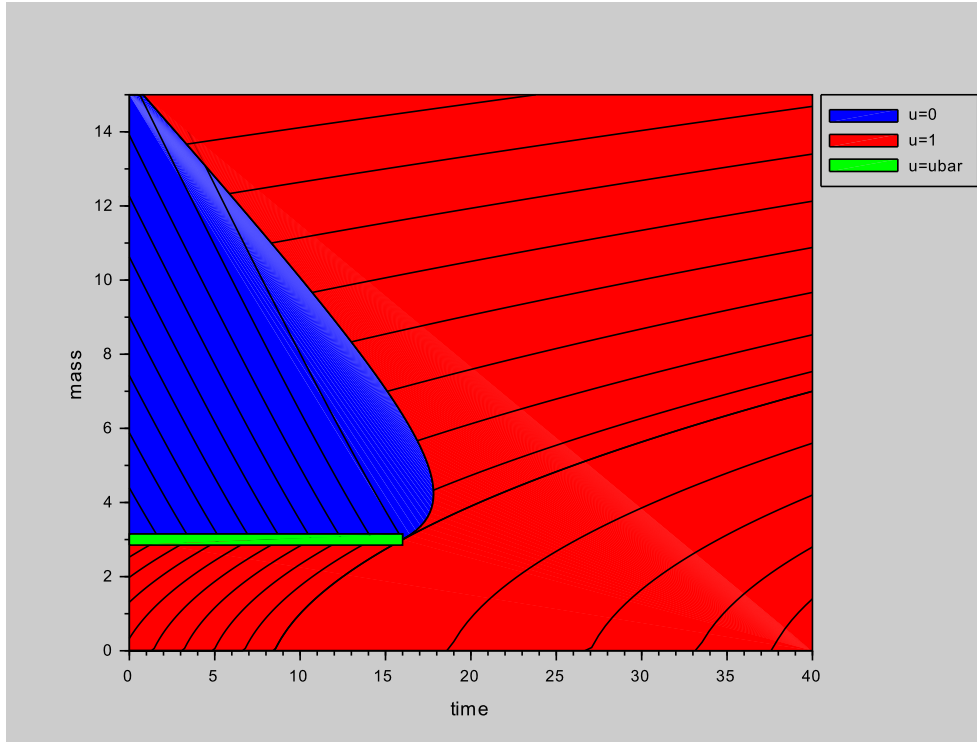


Figure 2: Model of Section 5.2: Synthesis for $a = b = e = 1$ and $T = 40$ hours; in red, $u = 1$, in blue $u = 0$, the horizontal line which separates them corresponds to \bar{u} while the curve connected to this line and which separates the red and blue regions on the right of the figure is the “switching curve”

6 Conclusion

In this work, the application of the Pontryagin’s Maximum Principle for the synthesis of the optimal control of a switched system showed interesting results for maximizing the treated volume of water in a filtration system. The optimal synthesis exhibits bang-bang controls with a “most rapid approach” to a singular arc and a switching curve before reaching the final time.

The main advantage of the optimal control approach proposed here is that it has been synthesized for a very large class of models, essentially defined by qualitative properties of functions f_1 , f_2 and g .

Perspectives of this work is to study the practical implementation of the optimal synthesis with real process constraints, and then to compare the water production of the membrane filtration process with the classical operating strategy proposed in the literature.

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References

- [1] Benyahia, B., Charfi, A., Benamar, N., Héran, M., Grasmick, A., Cherki, B., and Harmand, J.(2013). *A simple model of anaerobic membrane bioreactor for control design: coupling the “AM2b” model with a simple membrane fouling dynamics*. 13. *World Congress on Anaerobic Digestion: Recovering (bio) Resources for the World*. AD13, 171–p.

- [2] Boscain, U. and Piccoli, B. *Optimal Syntheses for Control Systems on 2-D Manifolds*, Springer SMAI, vol. 43, 2005.
- [3] Cogan, N.G. and Chellamb, S. (2014) *A method for determining the optimal back-washing frequency and duration for dead-end microfiltration*. *Journal of Membrane Science*, 469, 410-417.
- [4] Cogan, N.G., Li, J., Badireddy, A.R. and Chellamb, S. (2016) *Optimal backwashing in dead-end bacterial microfiltration with irreversible attachment*. *Journal of Membrane Science*, 520, 337-344.
- [5] Ferrero, G., Monclús, H., Buttiglieri, G., Comas, J. and Rodríguez-Roda, I. (2011). *Automatic control system for energy optimization in membrane bioreactors*. *Desalination*, 268(1), 276-280.
- [6] Ferrero, G. Rodríguez-Roda, I. and Comas, J. (2012). Automatic control systems for submerged membrane bioreactors: A state-of-the-art review. *water research, Elsevier*, 46, 3421–3433.
- [7] Kalboussi, N., Harmand, J., Ben Amar, N. and Ellouze, F. (2016). A comparative study of three membrane fouling models - Towards a generic model for optimization purposes. CARI'2016, 10-16 october, Tunis, Tunisia.
- [8] Hong, S. N., Zhao, H. W. and DiMassimo, R. W. (2008). *A new approach to backwash initiation in membrane systems*. U.S. Patent No. 7,459,083. Washington, DC: U.S. Patent and Trademark Office.
- [9] Lee, E.B. and Markus, L. (1967) *Foundations of Optimal Control Theory*. SIAM series in applied mathematics, Wiley.
- [10] Pontryagin, L.S., Boltyanskiy, V.G., Gamkrelidze, R.V. and Mishchenko, E.F. (1964) *Mathematical Theory of Optimal Processes*, The Macmillan Company.
- [11] Robles, A., Ruano, M. V., Ribes, J. and Ferrero, J. (2013). *Advanced control system for optimal filtration in submerged anaerobic MBRs (SAnMBRs)*. *Journal of Membrane Science*, 430, 330-341.
- [12] Smith, P. J., Vigneswaran, S., Ngo, H. H., Ben-Aim, R. and Nguyen, H. (2006). *A new approach to backwash initiation in membrane systems*. *Journal of Membrane Science*, 278(1), 381-389.
- [13] Vargas, A., Moreno-Andrade, I. and Buitrón, G. (2008). *Controlled backwashing in a membrane sequencing batch reactor used for toxic wastewater treatment*. *Journal of Membrane Science*, 320(1), 185-190.